DPMS: Advanced Materials

Lecture E11:
Computational Optics II

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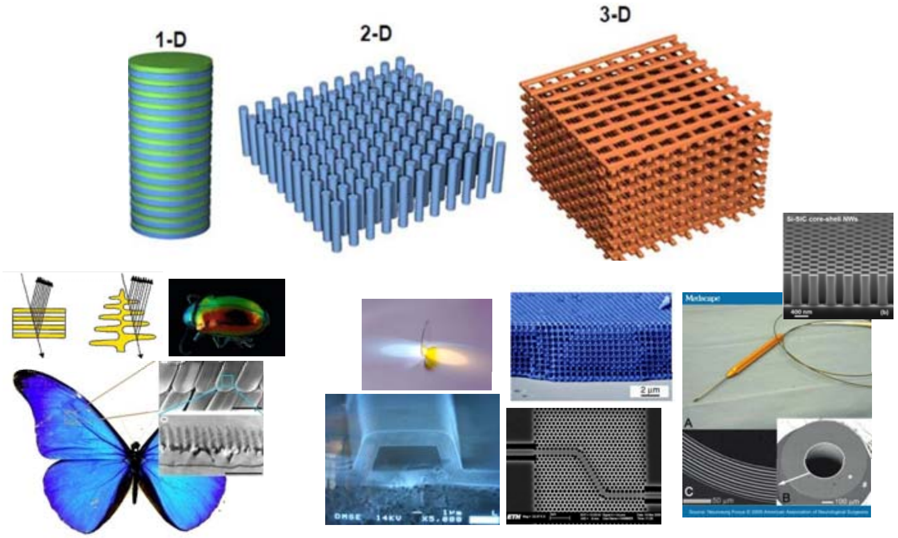
Photonic crystals in technology

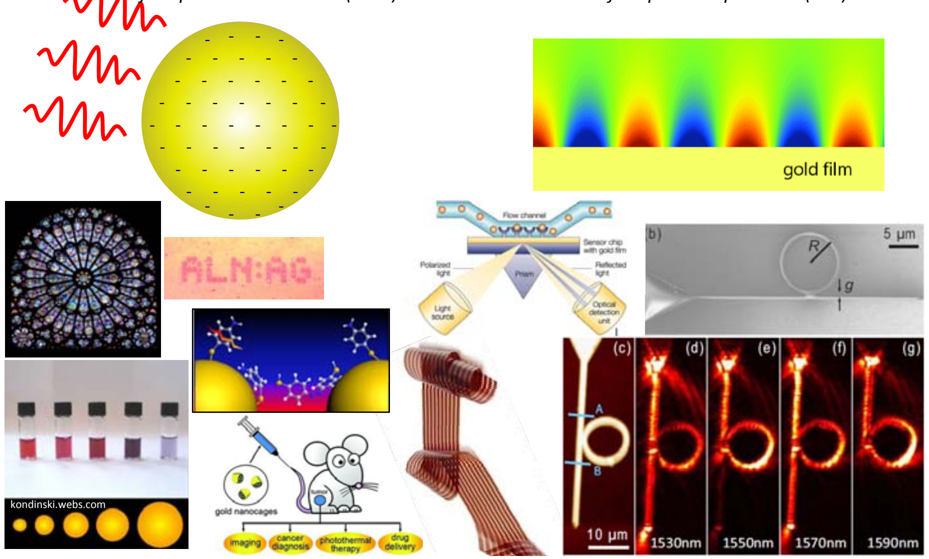
- Applications in extreme light manipulation



Plasmonics in technology

Localized Plasmons
localized surface plasmon resonance (LSPR)

Propagating Plasmons
surface plasmon polariton (SPP)

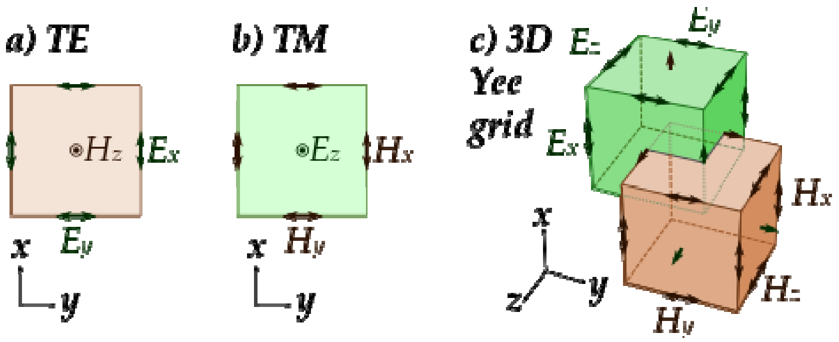


Numerical solutions in photonics

- Time domain
  - direct time integration of Maxwell's equations
  - main tool the finite-difference time-domain (FDTD) method
  - frequency information obtained by Fourier transform of the time evolution
- Frequency domain
  - eigenvalue solutions
  - plane-wave expansion method, finite element method, transfer matrix method
  - time domain information by solution superposition

Time domain

- Yee lattice: different sublattices for electric and magnetic fields
  - central differences for all field components



E/M waves in 3D

- E/M waves in matter

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = +\frac{\partial \mathbf{D}}{\partial t}$$

constitutive relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E} \qquad \mathbf{B} = \mu_0 \mathbf{H}$$

dielectric function

$$\varepsilon_r = \varepsilon_\infty + \sum_j \frac{f_j^2}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

  - ε_∞ is a constant polarizability, coming from electronic orbitals of much higher frequencies

- How do we program frequency dependent ε_r into the time domain equations?
  - we use the polarizability differential equation

$$\frac{\partial^2 \mathbf{P}_j}{\partial t^2} + \gamma_j \frac{\partial \mathbf{P}_j}{\partial t} + \omega_j^2 \mathbf{P}_j = \varepsilon_0 f_j^2 \mathbf{E}$$

  - indeed, if we combine this with $\mathbf{E}, \mathbf{P}_j \propto e^{-i\omega t}$, $\mathbf{P} = \sum_j \mathbf{P}_j$ and $\varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E}$, we find

$$-\omega^2 \mathbf{P}_j - i\omega\gamma_j \mathbf{P}_j + \omega_j^2 \mathbf{P}_j = \varepsilon_0 f_j^2 \mathbf{E}$$

$$\Rightarrow \mathbf{P}_j = \frac{\varepsilon_0 f_j^2}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_\infty \mathbf{E} + \sum_j \mathbf{P}_j = \varepsilon_0 \left(\varepsilon_\infty + \frac{f_j^2}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right) \mathbf{E}$$

E/M waves in 3D

- Expansion of the curl

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial_y} - \frac{\partial E_y}{\partial_z} \right) - \hat{y}\left(\frac{\partial E_z}{\partial_x} - \frac{\partial E_x}{\partial_z} \right) + \hat{z}\left(\frac{\partial E_y}{\partial_x} - \frac{\partial E_x}{\partial_y} \right)$$

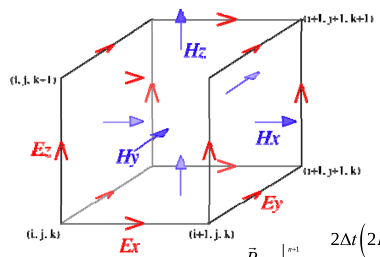
  - so Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = +\frac{\partial \mathbf{D}}{\partial t} \qquad \frac{\partial^2 \mathbf{P}_j}{\partial t^2} + \gamma_j \frac{\partial \mathbf{P}_j}{\partial t} + \omega_j^2 \mathbf{P}_j = \varepsilon_0 f_j^2 \mathbf{E}$$

$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t}$	$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon_0 \varepsilon_\infty \frac{\partial E_x}{\partial t} + \sum_j \frac{\partial P_{jx}}{\partial t}$	$\frac{\partial^2 P_{jx}}{\partial t^2} + \gamma_j \frac{\partial P_{jx}}{\partial t} + \omega_j^2 P_{jx} = \varepsilon_0 f_j^2 E_x$
$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_0 \frac{\partial H_y}{\partial t}$	$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon_0 \varepsilon_\infty \frac{\partial E_y}{\partial t} + \sum_j \frac{\partial P_{jy}}{\partial t}$	$\frac{\partial^2 P_{jy}}{\partial t^2} + \gamma_j \frac{\partial P_{jy}}{\partial t} + \omega_j^2 P_{jy} = \varepsilon_0 f_j^2 E_y$
$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t}$	$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_0 \varepsilon_\infty \frac{\partial E_z}{\partial t} + \sum_j \frac{\partial P_{jz}}{\partial t}$	$\frac{\partial^2 P_{jz}}{\partial t^2} + \gamma_j \frac{\partial P_{jz}}{\partial t} + \omega_j^2 P_{jz} = \varepsilon_0 f_j^2 E_z$

3D FDTD equations

- FDTD and the Yee lattice



$$\vec{H}_x\Big|_{i,j,k}^{n+3/2} = \vec{H}_x\Big|_{i,j,k}^{n+1/2} + \frac{\Delta t}{\mu_{i,j,k}} \left(\frac{\vec{E}_y\big|_{i,j,k+1/2}^{n+1} - \vec{E}_y\big|_{i,j,k-1/2}^{n+1}}{\Delta z} - \frac{\vec{E}_z\big|_{i,j+1/2,k}^{n+1} - \vec{E}_z\big|_{i,j-1/2,k}^{n+1}}{\Delta y} \right)$$

$$\vec{H}_y\Big|_{i,j,k}^{n+3/2} = \vec{H}_y\Big|_{i,j,k}^{n+1/2} + \frac{\Delta t}{\mu_{i,j,k}} \left(\frac{\vec{E}_z\big|_{i+1/2,j,k}^{n+1} - \vec{E}_z\big|_{i-1/2,j,k}^{n+1}}{\Delta x} - \frac{\vec{E}_x\big|_{i,j,k+1/2}^{n+1} - \vec{E}_x\big|_{i,j,k-1/2}^{n+1}}{\Delta z} \right)$$

$$\vec{H}_z\Big|_{i,j,k}^{n+3/2} = \vec{H}_z\Big|_{i,j,k}^{n+1/2} + \frac{\Delta t}{\mu_{i,j,k}} \left(\frac{\vec{E}_x\big|_{i,j+1/2,k}^{n+1} - \vec{E}_x\big|_{i,j-1/2,k}^{n+1}}{\Delta y} - \frac{\vec{E}_y\big|_{i+1/2,j,k}^{n+1} - \vec{E}_y\big|_{i-1/2,j,k}^{n+1}}{\Delta x} \right)$$

$$\vec{P}_{x,y,z}\Big|_{i,j,k}^{n+1} = \frac{2\Delta t\left(2\vec{P}_{x,y,z}\big|_{i,j,k}^{n} - \vec{P}_{x,y,z}\big|_{i,j,k}^{n-1}\right) + \Delta t^2\left(\gamma_0 \vec{P}_{x,y,z}\big|_{i,j,k}^{n-1} + 2\Delta t\left(\varepsilon\sigma_0{}^2 \mathrm{E}_{x,y,z}\big|_{i}^{n} - \omega_{0_0}^2 \vec{P}_{x,y,z}\big|_{i,j,k}^{n}\right)\right)}{2\Delta t + \gamma_0 \Delta t^2}$$

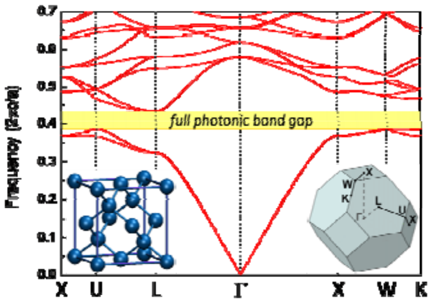
$$\vec{E}_x\Big|_{i,j,k}^{n+1} = \vec{E}_x\Big|_{i,j,k}^{n} + \vec{P}_x\Big|_{i,j,k}^{n+1} - \vec{P}_x\Big|_{i,j,k}^{n-1} + \frac{\Delta t}{\varepsilon_{i,j,k}} \left(\frac{\vec{H}_z\big|_{i,j+1/2,k}^{n+1/2} - \vec{H}_z\big|_{i,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{\vec{H}_y\big|_{i,j,k+1/2}^{n+1/2} - \vec{H}_y\big|_{i,j,k-1/2}^{n+1/2}}{\Delta z} \right)$$

$$\vec{E}_y\Big|_{i,j,k}^{n+1} = \vec{E}_y\Big|_{i,j,k}^{n} + \vec{P}_y\Big|_{i,j,k}^{n+1} - \vec{P}_y\Big|_{i,j,k}^{n-1} + \frac{\Delta t}{\varepsilon_{i,j,k}} \left(\frac{\vec{H}_x\big|_{i,j,k+1/2}^{n+1/2} - \vec{H}_x\big|_{i,j,k-1/2}^{n+1/2}}{\Delta z} - \frac{\vec{H}_z\big|_{i+1/2,j,k}^{n+1/2} - \vec{H}_z\big|_{i-1/2,j,k}^{n+1/2}}{\Delta x} \right)$$

$$\vec{E}_z\Big|_{i,j,k}^{n+1} = \vec{E}_z\Big|_{i,j,k}^{n} + \vec{P}_z\Big|_{i,j,k}^{n+1} - \vec{P}_z\Big|_{i,j,k}^{n-1} + \frac{\Delta t}{\varepsilon_{i,j,k}} \left(\frac{\vec{H}_y\big|_{i+1/2,j,k}^{n+1/2} - \vec{H}_y\big|_{i-1/2,j,k}^{n+1/2}}{\Delta x} - \frac{\vec{H}_x\big|_{i,j+1/2,k}^{n+1/2} - \vec{H}_x\big|_{i,j-1/2,k}^{n+1/2}}{\Delta y} \right)$$

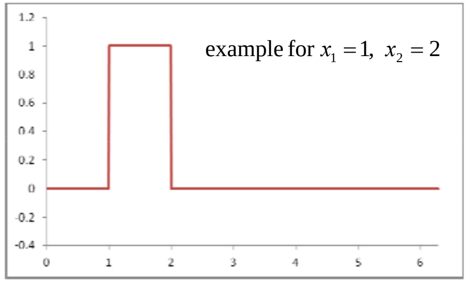
Frequency domain

- Eigensolutions for each frequency separately
  - determine ω *vs* k relation: photonic band structure
  - transport properties, band gaps, eigenmodes
- Fourier analysis: plane wave expansion method



Fourier transforms

- Assume the step-function $f(x)=\begin{cases}1 & x_1<x<x_2\\ 0 & x<x_1 \text{ or } x>x_2\end{cases}$
  - $0 < x < 2\pi$



- Can we write this as a sum of sines and cosines?

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

  - how do we find the constants a_n and b_n?

Fourier transforms

- We want the coefficients of this expansion

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

  - we take advantage of the orthogonality of the trigonometric functions

$$\int_0^{2\pi}\sin(nx)\sin(mx)dx=\begin{cases}\pi\delta_{nm} & n,m\neq 0\\ 0 & n=m=0\end{cases}\qquad \int_0^{2\pi}\sin(nx)\cos(mx)dx=0$$

$$\int_0^{2\pi}\cos(nx)\cos(mx)dx=\begin{cases}\pi\delta_{nm} & n,m\neq 0\\ 2\pi & n=m=0\end{cases}$$

- Multiply our expression with sin(mx) or cos(mx) and integrate

$$\int_0^{2\pi}\sin(mx)f(x)dx=\sum_{n=0}^{N}\left[\int_0^{2\pi}a_n\sin(nx)\sin(mx)dx+\int_0^{2\pi}b_n\cos(nx)\sin(mx)dx\right]=\begin{cases}\pi a_m & m>0\\ 0 & m=0\end{cases}$$

$$\int_0^{2\pi}\cos(mx)f(x)dx=\sum_{n=0}^{N}\left[\int_0^{2\pi}a_n\sin(nx)\cos(mx)dx+\int_0^{2\pi}b_n\cos(nx)\cos(mx)dx\right]=\begin{cases}\pi b_m & m>0\\ 2\pi b_0 & m=0\end{cases}$$

Fourier transforms

- Our expansion coefficients

$$a_m=\frac{1}{\pi}\int_0^{2\pi}\sin(mx)f(x)dx\qquad b_m=\frac{1}{\pi}\int_0^{2\pi}\cos(mx)f(x)dx\qquad a_0=0\qquad b_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx$$

- Calculate the expansion coefficients for our step function
  - the n=0 term

$$b_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx=\frac{1}{2\pi}\int_{x_1}^{x_2}dx=\frac{1}{2\pi}(x_2-x_1)=\frac{1}{2\pi}(2-1)=0.159155$$

  - the *n>0* terms

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\sin(nx)f(x)dx=\frac{1}{\pi}\int_{x_1}^{x_2}\sin(nx)dx=\frac{-1}{\pi n}\left[\cos(nx_2)-\cos(nx_1)\right]=\frac{\cos(n)-\cos(2n)}{\pi n}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\cos(nx)f(x)dx=\frac{1}{\pi}\int_{x_1}^{x_2}\cos(nx)dx=\frac{1}{\pi n}\left[\sin(nx_2)-\sin(nx_1)\right]=\frac{\sin(2n)-\sin(n)}{\pi n}$$

  - some first values

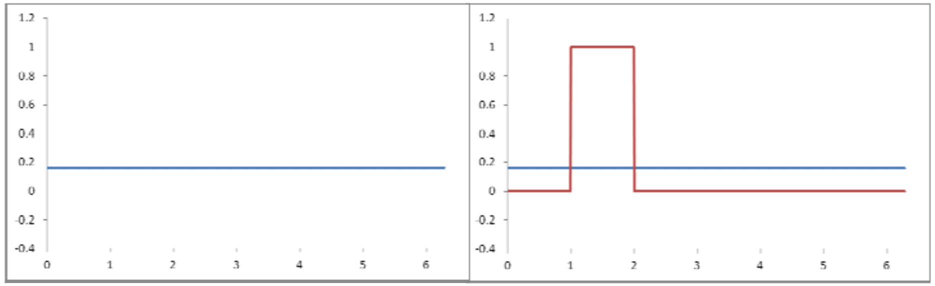
$$a_1=0.3044\qquad a_2=0.0378\qquad a_3=-0.2069\qquad \cdots$$

$$b_1=0.0216\qquad b_2=-0.2651\qquad b_3=-0.0446\qquad \cdots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=0

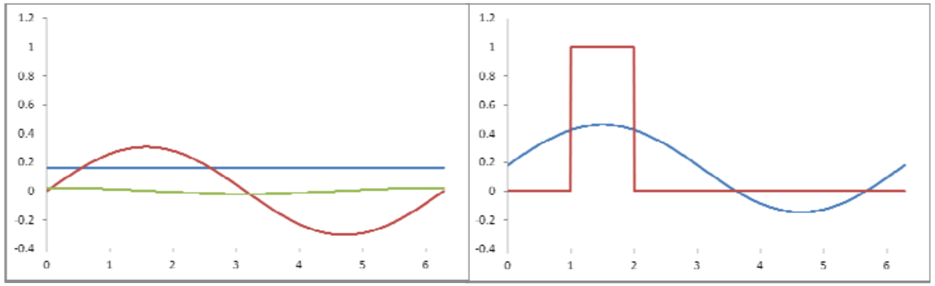


$$f(x)=b_0$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=1

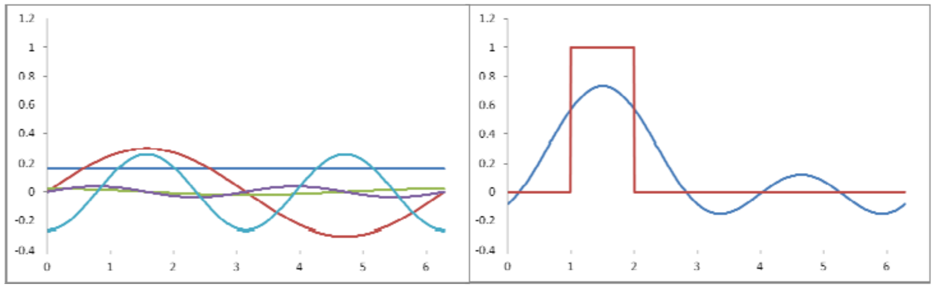


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=2

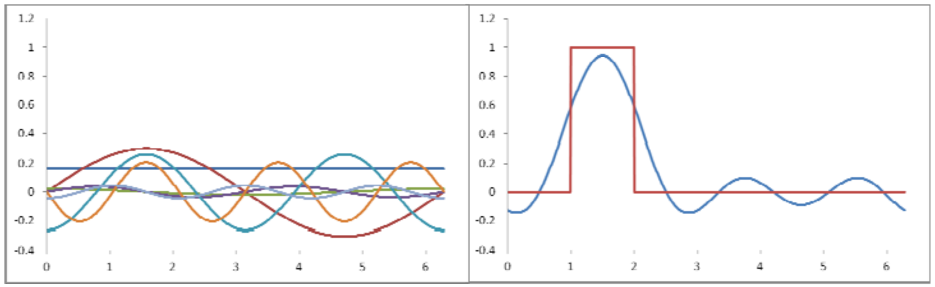


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=3

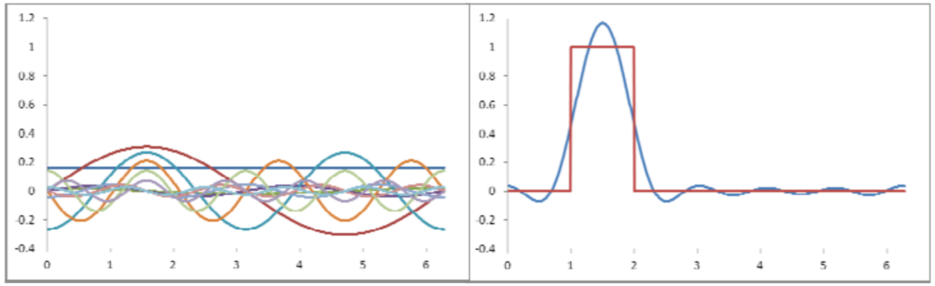


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n \sin(nx)+b_n \cos(nx)\right]$$

N=5

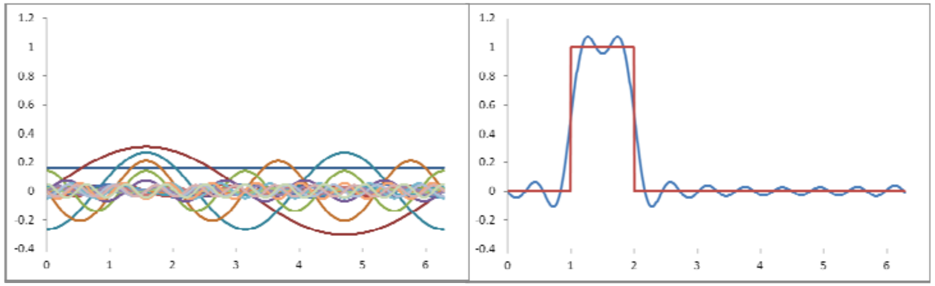


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n \sin(nx)+b_n \cos(nx)\right]$$

N=10

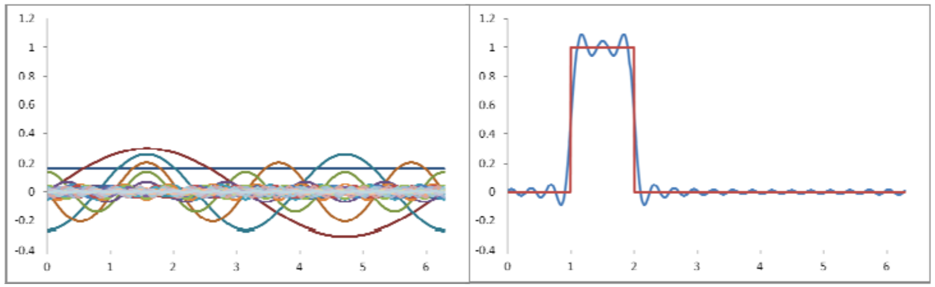


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n \sin(nx)+b_n \cos(nx)\right]$$

N=20

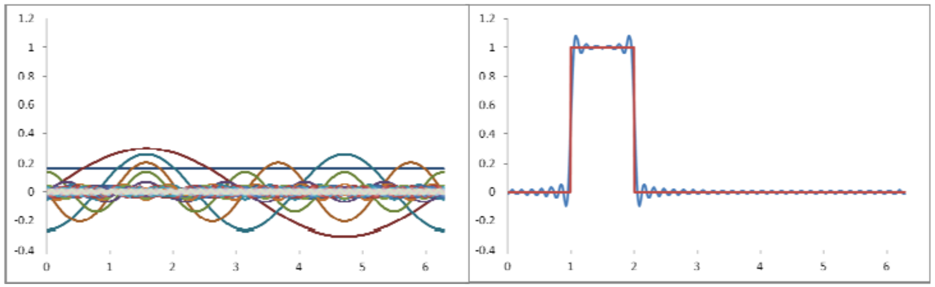


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n \sin(nx)+b_n \cos(nx)\right]$$

N=40

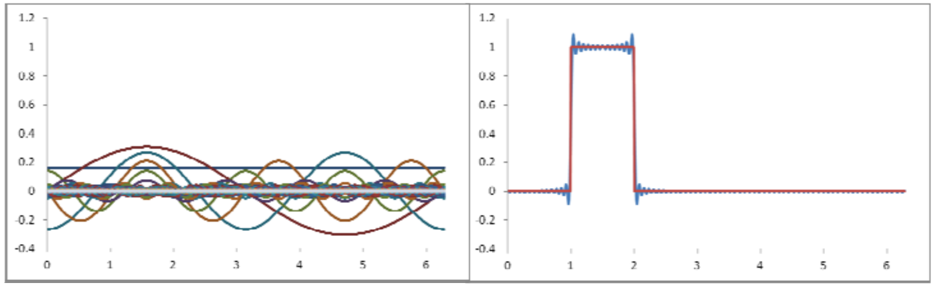


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=80

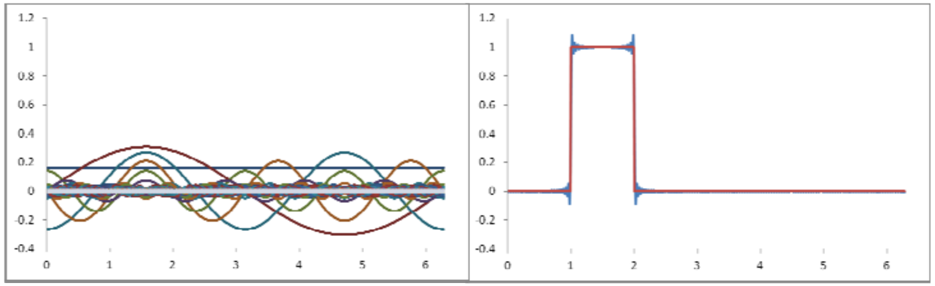


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Fourier transform

$$f(x)=\sum_{n=0}^{N}\left[a_n\sin(nx)+b_n\cos(nx)\right]$$

N=160

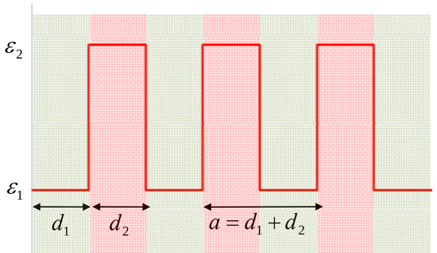


$$f(x)=b_0+a_1\sin(x)+b_1\cos(x)+a_2\sin(2x)+b_2\cos(2x)+\ldots$$

Plane wave expansion method

- Can we apply the Fourier method to find the solutions of a photonic crystal?
  - photonic crystal in 1D

$$\varepsilon(x)=\begin{cases}\varepsilon_1 & 0<x<d_1\\ \varepsilon_2 & d_1<x<a\end{cases}$$



  - wave equation in 1D

$$\frac{d^2E(x)}{dx^2}=-\varepsilon(x)\frac{\omega^2}{c^2}E(x)$$

  - can we write the field and dielectric function as plane wave expansions?

- The periodicity is critical
  - the dielectric constant is periodic, with period *a* $\quad \varepsilon(x+a)=\varepsilon(x)$
  - any plane wave used must also be periodic to *a*

$$e^{iq(x+a)}=e^{iqx}\quad\Rightarrow e^{iqa}=1\quad\Rightarrow q=\pm\frac{2\pi n}{a}$$

- We define the reciprocal lattice vectors $G_{\pm n}=\pm\frac{2\pi n}{a}$

Plane wave expansion method

- The expansion for the dielectric function

$$\varepsilon(x)=\sum_{n=-\infty}^{\infty}\varepsilon_n e^{iG_nx}$$

  - to find the coefficients we note that

$$\int_0^a e^{iG_nx}e^{iG_mx}dx=\int_0^a e^{i(2\pi n/a)x}e^{i(2\pi m/a)x}dx=\int_0^a e^{i(2\pi(n+m)/a)x}dx=\frac{a}{2\pi(n+m)}\int_0^{2\pi(n+m)}e^{iy}dy=\begin{cases}0 & n\neq -m\\ a & n=-m\end{cases}$$

  - so $\quad \varepsilon_n=\frac{1}{a}\int_0^a e^{-iG_nx}\varepsilon(x)dx$

- The expansion for the electric field
  - Bloch's theorem: the solution is the product of a periodic function times a plane wave

$$E_q(x)=\sum_{n=-\infty}^{\infty}E_{q,n}e^{i(q+G_n)x}$$

  - the solution involves finding the $E_{k,n}$ for each *k* and *n*

- For each k we will find the allowed frequencies
  - frequency domain solution: standing waves, band structure

Plane wave expansion method

- For the solution
  - we substitute $\varepsilon(x)=\sum_{n=-\infty}^{\infty}\varepsilon_n e^{iG_n x}$ $\quad E_q(x)=\sum_{n=-N}^{N}E_{q,n}e^{i(q+G_n)x}$ into $\frac{d^2E(x)}{dx^2}=-\varepsilon(x)\frac{\omega^2}{c^2}E(x)$
  - then the left side
  $$\frac{d^2}{dx^2}E_q(x)=\sum_{n=-\infty}^{\infty}E_{q,n}\frac{d^2}{dx^2}e^{i(q+G_n)x}=-\sum_{n=-\infty}^{\infty}E_{q,n}|q+G_n|^2e^{i(q+G_n)x}$$
  - and the right side
  $$-\varepsilon(x)\frac{\omega^2}{c^2}E_q(x)=-\frac{\omega^2}{c^2}\sum_{n''=-\infty}^{\infty}\varepsilon_{n''}e^{iG_{n''}x}\sum_{n'=-\infty}^{\infty}E_{q,n'}e^{i(q+G_{n'})x}=-\frac{\omega^2}{c^2}\sum_{n''=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\varepsilon_{n''}e^{iG_{n''}x}E_{q,n'}e^{i(q+G_{n'}+G_{n''})x}$$
- We multiply and integrate both parts with $e^{-i(q+G_m)x}$
  - the left side $\quad =-\sum_{n=-\infty}^{\infty}E_{q,n}|q+G_n|^2\int e^{-i(q+G_m)x}e^{i(q+G_n)x}dx \quad = aE_{q,m}|q+G_m|^2$
  - the right side $\quad \frac{\omega^2}{c^2}\sum_{n''=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\varepsilon_{n''}E_{q,n'}\int e^{-i(q+G_m)x}e^{i(q+G_{n'}+G_{n''})x}dx$
  - the only terms surviving are the ones with $\quad m=n'+n''\Rightarrow n''=m-n'$
- The eigenvalue problem $\quad E_{q,m}|q+G_m|^2=\frac{\omega^2}{c^2}\sum_{n'=-\infty}^{\infty}\varepsilon_{m-n'}E_{q,n'}$

Plane wave expansion method

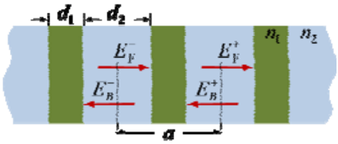
- Our equation is
$$E_{q,m}|q+G_m|^2=\frac{\omega^2}{c^2}\sum_{n'=-\infty}^{\infty}\varepsilon_{m-n'}E_{q,n'}$$

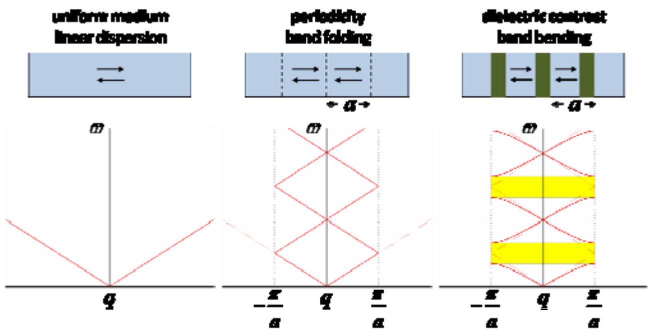
$$\begin{pmatrix}\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots\\ \cdots & |q+G_{-2}|^2 & 0 & 0 & 0 & 0 & \cdots\\ \cdots & 0 & |q+G_{-1}|^2 & 0 & 0 & 0 & \cdots\\ \cdots & 0 & 0 & |q+G_0|^2 & 0 & 0 & \cdots\\ \cdots & 0 & 0 & 0 & |q+G_1|^2 & 0 & \cdots\\ \cdots & 0 & 0 & 0 & 0 & |q+G_2|^2 & \cdots\\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\vdots\\ E_{q,-2}\\ E_{q,-1}\\ E_{q,0}\\ E_{q,1}\\ E_{q,2}\\ \vdots\end{pmatrix}=\frac{\omega^2}{c^2}\begin{pmatrix}\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots\\ \cdots & \varepsilon_0 & \varepsilon_{-1} & \varepsilon_{-2} & \varepsilon_{-3} & \varepsilon_{-4} & \cdots\\ \cdots & \varepsilon_1 & \varepsilon_0 & \varepsilon_{-1} & \varepsilon_{-2} & \varepsilon_{-3} & \cdots\\ \cdots & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & \varepsilon_{-1} & \varepsilon_{-2} & \cdots\\ \cdots & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & \varepsilon_{-1} & \cdots\\ \cdots & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & \varepsilon_0 & \cdots\\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\vdots\\ E_{q,-2}\\ E_{q,-1}\\ E_{q,0}\\ E_{q,1}\\ E_{q,2}\\ \vdots\end{pmatrix}$$

  - this is a matrix equation
  $$\mathbf{G}\cdot\mathbf{E}=\frac{\omega^2}{c^2}\boldsymbol{\varepsilon}\cdot\mathbf{E}\ \Rightarrow\ \boldsymbol{\varepsilon}^{-1}\cdot\mathbf{G}\cdot\mathbf{E}=\frac{\omega^2}{c^2}\mathbf{E}$$
  - i.e. of the type $\quad \mathbf{M}\cdot\mathbf{X}=\lambda\mathbf{X}$
- The eigenvalues and eigenvectors are found by standard diagonilization of **M**
  - for each *q* we find multiple eigenfrequencies *ω/c*

Photonic band structure

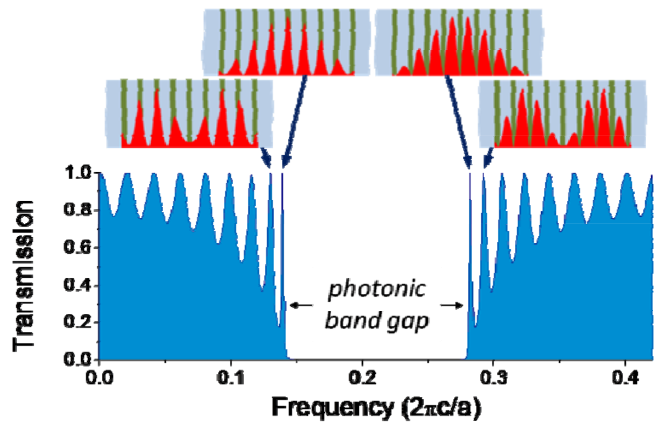
- The *q* is periodic to $2\pi/a$
  - $q+2\pi/a$ and *q* are identical
  - dispersion curves are folded inside the Brillouin zone
  - if there is scattering, band gaps open at zone edges





Photonic band gaps

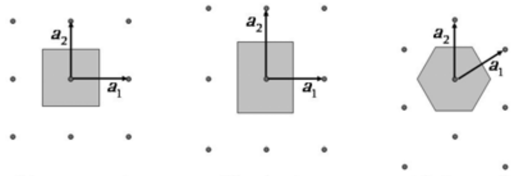
- We can also calculate the shape of the eigenvectors (standing waves)



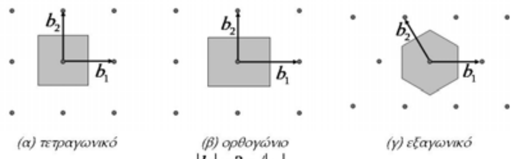
Plane wave expansion in 2D and 3D

- For simplicity we look at 2D lattices
  - we first define the unit cell and the lattice vectors a_1 and a_2
  - as in 1D, periodicity is important

square lattice *orthogonal lattice* *hexagonal lattice*



- next we define the reciprocal lattice vectors (i.e. our 2D **G** vectors)



$|b_1| = |b_2| = 2\pi/|a_1|$ $\quad |b_1| = 2\pi/|a_1|$, $|b_2| = 2\pi/|a_2|$ $\quad |b_1| = |b_2| = 4\pi/\sqrt{3}|a_1|$

$$\mathbf{b}_1 = \frac{2\pi(\mathbf{a}_2 \times \mathbf{a}_3)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}$$

$$\mathbf{b}_2 = \frac{2\pi(\mathbf{a}_3 \times \mathbf{a}_1)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}$$

$$\mathbf{b}_3 = \frac{2\pi(\mathbf{a}_1 \times \mathbf{a}_2)}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}$$

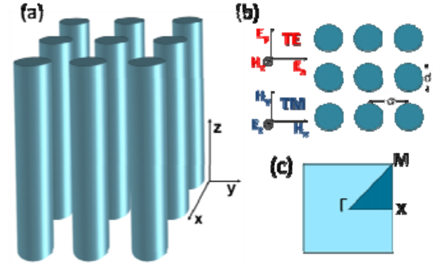
2D photonic crystals

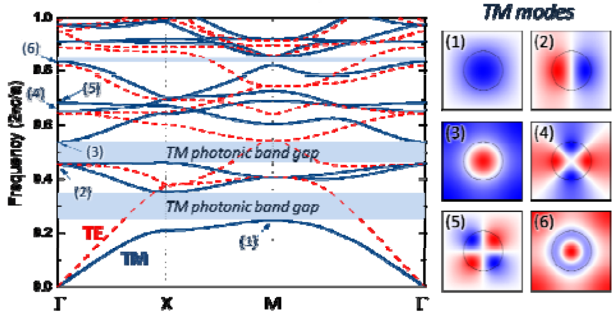
- Eigenvalue problem

$$|\mathbf{k}+\mathbf{G}| E_{\mathbf{G}} = \frac{\omega^2}{c^2} \sum_{G'} \varepsilon_{\mathbf{G}-\mathbf{G}'} E_{\mathbf{G}'} \qquad \text{or,}$$

$$\sum_{\mathbf{G}'} \varepsilon^{-1}_{\mathbf{G}-\mathbf{G}'} (\mathbf{k}+\mathbf{G})(\mathbf{k}+\mathbf{G}') H_{\mathbf{G}'} = \frac{\omega^2}{c^2} H_{\mathbf{G}}$$

$$\mathbf{G} = n_1 \mathbf{b}_1 + n_2 \mathbf{b}_2$$





2D and 3D photonic systems

- Various 2D and 3D applications
  - Guided light
  - Trapped light